Lecture IX: Bogoliubov Theory of weakly interacting Bose gas

Although strong interaction effects can lead to the formation of novel ground states of the electron system, the properties of the weakly interacting system mirror closely the trivial behaviour of the non-interacting fermi gas. By contrast, even in the weakly interacting system, the Bose gas has the capacity to form a Bose-Einstein condensed phase. The aim of this lecture is to explore the nature of the ground state and the character of the elementary excitation spectrum in the condensed phase.

▷ WEAKLY INTERACTING BOSE GAS

Consider a system of N bosons confined to a volume L^d

In the non-interacting system, at $T = 0$, all bosons are condensed in the lowest energy single-particle state, viz. $|\text{g.s.}\rangle = \frac{1}{N!}(a_0^\dagger)^N|\Omega\rangle$

Aim: How is ground state and spectrum of elementary excitations influenced by weak interaction?

$$\hat{H} = \sum_{\mathbf{k}} \frac{\hbar^2\mathbf{k}^2}{2m} a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \overbrace{\frac{1}{2}\int d^dx \int d^dx'\, a^\dagger(\mathbf{x})a^\dagger(\mathbf{x}')V(\mathbf{x}-\mathbf{x}')a(\mathbf{x}')a(\mathbf{x})}^{\hat{H}_I}$$

$V(\mathbf{x}-\mathbf{x}')$ — pairwise particle interaction

In Fourier basis, with $a(\mathbf{x}) = \frac{1}{L^{d/2}}\sum_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} a_{\mathbf{k}}$ and $V(\mathbf{q}) = \int d^dx\, e^{-i\mathbf{q}\cdot\mathbf{x}} V(\mathbf{x})$

$$\hat{H}_I = \frac{1}{2L^d}\sum_{\mathbf{k},\mathbf{k}',\mathbf{q}} V(\mathbf{q})\, a_{\mathbf{k}+\mathbf{q}}^\dagger a_{\mathbf{k}'}^\dagger a_{\mathbf{k}} a_{\mathbf{k}'+\mathbf{q}}$$

If interaction is weak, in condensed phase, one may assume that the lowest-lying single-particle state is still macroscopically occupied, i.e. $N_0/N = \mathcal{O}(1)$

Therefore, since $\hat{N}_0 = a_{k=0}^\dagger a_{k=0} = \mathcal{O}(N) \gg 1$ and $a_0 a_0^\dagger - a_0^\dagger a_0 = 1$, to a good approximation, a_0 and $a_0^\dagger$ can be replaced by the ordinary c-number $\sqrt{N_0}$

Taking (for simplicity) $V(\mathbf{q}) = V$ const., i.e. a contact interaction, expansion in N_0 obtains

$$\hat{H}_I = \frac{V}{2L^d}N_0^2 + \frac{V}{L^d}N_0\sum_{\mathbf{k}\neq 0}\left[a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger a_{-\mathbf{k}} + \frac{1}{2}\left(a_{-\mathbf{k}}a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger a_{-\mathbf{k}}^\dagger\right)\right] + O(N_0^0)$$

cf. quantum AF in spin-wave approximation

▷ Physical interpretation of components:

- $V a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ represents the 'Hartree-Fock energy' of excited particles interacting with condensate — *Note that the contact nature of the interaction disguises the presence of the direct and exchange contributions*

- $V(a_{-\mathbf{k}}a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}})$ represents creation or annihilation of excited particles from the condensate; *Note that, in this approximation, total no. of particles is not conserved*

Finally, using the identity $N = N_0 + \sum_{\mathbf{k}\neq 0} a^\dagger_{\mathbf{k}} a_{\mathbf{k}}$ to trade N_0 for N,

$$\hat{H} = \frac{VnN}{2} + \sum_{\mathbf{k}\neq 0}\left[\left(\epsilon^0_{\mathbf{k}} + Vn\right) a^\dagger_{\mathbf{k}}a_{\mathbf{k}} + \frac{Vn}{2}\left(a_{-\mathbf{k}}a_{\mathbf{k}} + a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}}\right)\right]$$

where $n = N/L^d$ denotes number density and $\epsilon^0_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$

As with quantum AF, $\hat{H}$ diagonalised by Bogoluibov transformation:

$$\begin{pmatrix} a_{\mathbf{k}} \\ a^\dagger_{-\mathbf{k}} \end{pmatrix} = \begin{pmatrix} \cosh\theta_{\mathbf{k}} & -\sinh\theta_k \\ -\sinh\theta_{\mathbf{k}} & \cosh\theta_{\mathbf{k}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathbf{k}} \\ \alpha^\dagger_{-\mathbf{k}} \end{pmatrix}$$

Left as exercise to show that, when $\tanh(2\theta_k) = Vn/(\epsilon^0_{\mathbf{k}} + Vn)$,

$$\hat{H} = \frac{VnN}{2} - \frac{1}{2}\sum_{\mathbf{k}\neq 0}(\epsilon^0_{\mathbf{k}} + nV) + \sum_{\mathbf{k}\neq 0} \overbrace{\left[\left(\epsilon^0_{\mathbf{k}} + Vn\right)^2 - (Vn)^2\right]^{1/2}}^{\epsilon_{\mathbf{k}}} \left(\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \frac{1}{2}\right)$$

In particular, for $|\mathbf{k}| \to 0$, spectrum of low-energy excitations scales as $\epsilon(\mathbf{k}) \simeq \hbar c|\mathbf{k}|$ with $c = (Vn/m)^{1/2}$

At high energies ($k > k_0 = mc/\hbar$), spectrum becomes free particle-like

▷ †Ground state wavefunction: defined by condition $\alpha_{\mathbf{k}}|\text{g.s.}\rangle$

Since Bogoluibov transformation can be written as $\alpha_{\mathbf{k}} = U a_{\mathbf{k}} U^{-1}$ where (exercise)

$$U = \exp\left[\sum_{\mathbf{k}\neq 0}\frac{\theta_{\mathbf{k}}}{2}(a^\dagger_{\mathbf{k}}a^\dagger_{-\mathbf{k}} - a_{\mathbf{k}}a_{-\mathbf{k}})\right]$$

may infer g.s. $|\Phi_V\rangle$ from non-interacting g.s. $|\Phi_0\rangle$ as $|\Phi_V\rangle = U|\Phi_0\rangle$

Proof: since, for $V = 0$, all particles are in the $\mathbf{k} = 0$ state,

$$0 = a_{\mathbf{k}\neq 0}|\Phi_0\rangle = U^{-1}\,\overbrace{U a_{\mathbf{k}} U^{-1}}^{\alpha_{\mathbf{k}}}\,U|\Phi_0\rangle$$

▷ †Depletion of condensate due to interaction

$$\frac{N - N_0}{N} = \frac{1}{N}\sum_{\mathbf{k}\neq 0}\langle\text{g.s.}|a^\dagger_{\mathbf{k}}a_{\mathbf{k}}|\text{g.s.}\rangle = \frac{1}{N}\sum_{\mathbf{k}\neq 0}\sinh^2\theta_{\mathbf{k}} = \frac{1}{n}\int\frac{d^dk}{(2\pi)^3}\sinh^2\theta_{\mathbf{k}} \overset{\text{exercise}}{=} \frac{1}{3\pi^2 n}k_0^3$$

i.e. ca. one particle per ‘coherence length’ $\xi \sim 1/k_0$

Recast using scattering length (cf. TP2) of contact interaction $V = 4\pi\hbar^2 a/m$,

$$\frac{N - N_0}{N} = \frac{8}{3\sqrt{\pi}}(na^3)^{1/2}$$

▷ †Ground state energy

$$E_0 = \frac{VnN}{2} - \frac{1}{2}\sum_{\mathbf{k}\neq 0}(\epsilon^0_{\mathbf{k}} + nV - \epsilon_{\mathbf{k}} - \frac{(nV)^2}{2\epsilon^0_{\mathbf{k}}})$$

(where extra term controls unphysical UV divergence required by contact nature of potential)

$$\frac{E_0}{L^d} = \frac{n^2 V}{2}\left[1 + \frac{128}{15\sqrt{\pi}}(na^3)^{1/2}\right]$$

▷ Experiment? `transparencies`

When cooled to $T \sim 2K$, liquid ^{4}He undergoes transition to Bose-Einstein condensed state

Neutron scattering measurements can be used to infer spectrum of collective excitations

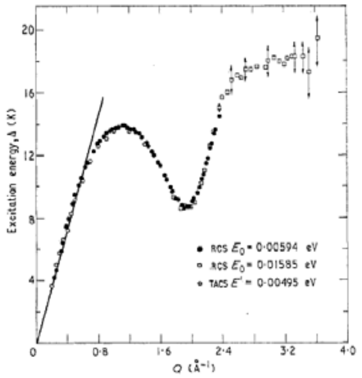


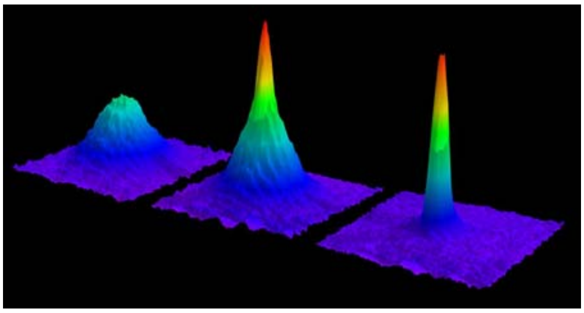
Figure 9. The dispersion curve for He II for the elementary excitations at $T = 1{\cdot}1$ K and $P \simeq 0$. Data obtained by rotating crystal spectrometer (RCS) and triple axis crystal spectrometer (TACS) as indicated. From Cowley and Woods (1971).

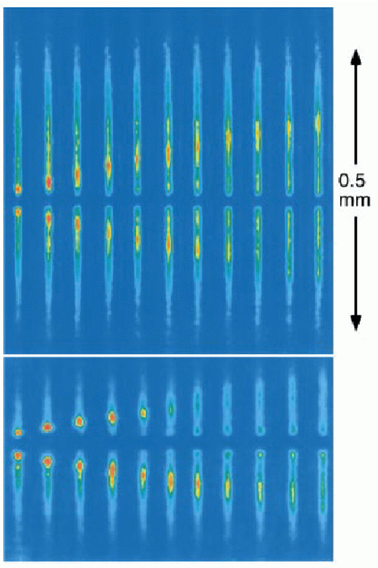
In Helium, steric interactions are strong and at higher energy scales an important second branch of excitations known as rotons appear

A second example of BEC is presented by ultracold atomic gases:

By confining atoms to a magnetic trap, time of flight measurements can be used to monitor momentum distribution of condensate

Moreover, the perturbation imposed by a laser due to the optical dipole interaction provides a means to measure the sound wave velocity





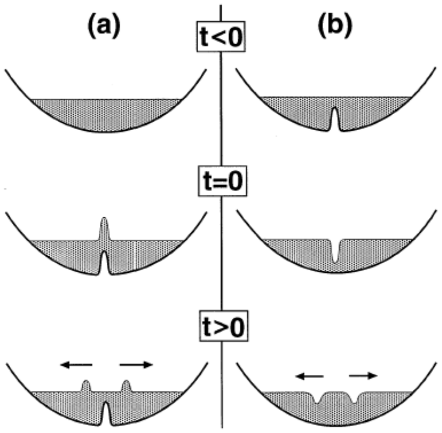


FIG. 1. Excitations of wave packets in a Bose-Einstein condensate. A condensate is confined in the potential of a magnetic trap. At time $t = 0$, a focused, blue-detuned laser beam is suddenly switched on (a) or off (b) and, by the optical dipole force, creates, respectively, two positive or negative pertubations in density which propagate at the speed of sound.

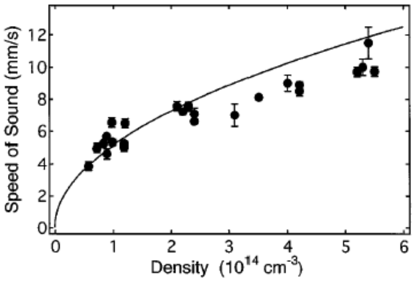


FIG. 4. Speed of sound versus condensate peak density. The solid line is the speed of sound [Eq. (2)] using the maximum cloud density [Eq. (4)] with no adjustable parameter. The error bars show only the statistical error.